

ON THE ANTI-AUTOMORPHISM OF THE STEENROD ALGEBRA: II

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ABSTRACT. The relations of Barratt and Miller are shown to include all relations among the elements $P^i\chi P^{n-i}$ in the mod p Steenrod algebra, and a minimal set of relations is given.

1. INTRODUCTION

Milnor [4] observed that the mod 2 Steenrod algebra \mathcal{A} forms a Hopf algebra with commutative diagonal determined by

$$(1) \quad \Delta \text{Sq}^n = \sum_i \text{Sq}^i \otimes \text{Sq}^{n-i}.$$

This allowed him to interpret the Cartan formula as the assertion that the cohomology of a space forms a module-algebra over \mathcal{A} . The anti-automorphism χ in the Hopf algebra structure, defined inductively by

$$(2) \quad \chi \text{Sq}^0 = \text{Sq}^0 \quad , \quad \sum_i \text{Sq}^i \chi \text{Sq}^{n-i} = 0 \quad \text{for } n > 0,$$

has a topological interpretation too: If K is a finite complex then the homology of the Spanier-Whitehead dual DK_+ of K_+ is canonically isomorphic to the cohomology of K . Under this isomorphism the left action by $\theta \in \mathcal{A}$ on $H^*(K)$ corresponds to the right action of $\chi\theta \in \mathcal{A}$ on $H_*(DK_+)$.

In 1974 Davis [3] proved that sometimes much more efficient ways exist to compute χSq^n :

$$(3) \quad \chi \text{Sq}^{2^r-1} = \text{Sq}^{2^r-1} \chi \text{Sq}^{2^r-1-1}$$

$$(4) \quad \chi \text{Sq}^{2^r-r-1} = \text{Sq}^{2^r-1} \chi \text{Sq}^{2^r-1-r} + \text{Sq}^{2^r-1} \chi \text{Sq}^{2^r-1-r-1}$$

Similarly, Straffin [5] proved that if $r \geq 0$ and $b \geq 2$ then

$$(5) \quad \sum_i \text{Sq}^{2^r i} \chi \text{Sq}^{2^r(b-i)} = 0.$$

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Both authors give analogous identities among reduced powers and their images under χ at an odd prime as well.

Barratt and Miller [1] found a general family of identities which includes all these as special cases. We state it for the general prime. When $p = 2$, P^n denotes Sq^n . Let $\alpha(n)$ denote the sum of the p -adic digits of n .

Theorem 1.1. [1, 2] *For any integer k and any integer $l \geq 0$ such that $pl - \alpha(l) < (p-1)n$*

$$(6) \quad \sum_i \binom{k-i}{l} P^i \chi P^{n-i} = 0.$$

The defining relations occur with $l = 0$. Davis's formulas (for $p = 2$) are the cases in which $(n, l, k) = (2^r - 1, 2^{r-1} - 1, 2^r - 1)$ or $(n, l, k) = (2^r - r - 1, 2^{r-1} - 2, 2^r - 2)$. Straffin's identities (for $p = 2$) occur as $(n, l, k) = (2^r b, 2^r - 1, 2^r b)$.

Since $\binom{(k+1)-i}{l} - \binom{k-i}{l} = \binom{k-i}{l-1}$, the cases $(l, k+1)$ and (l, k) of (6) imply it for $(l-1, k)$. Thus the relations for

$$(7) \quad l = l(n) = \max\{j : pj - \alpha(j) < (p-1)n\}$$

imply all the rest. When $p = 2$, $l(2^r - 1) = 2^{r-1} - 1$ and $l(2^r - r - 1) = 2^{r-1} - 2$, so Davis's relations are among these basic relations.

Two questions now arise. To express them uniformly in the prime, let \mathcal{P} denote the algebra of Steenrod reduced powers (which is the full Steenrod algebra when $p = 2$), but assign P^n degree n . It is natural to ask:

- Are there yet other linear relations among the $n+1$ elements $P^i \chi P^{n-i}$ in \mathcal{P}^n ?
- What is a minimal spanning set for

$$V_n = \text{Span}\{P^i \chi P^{n-i} : 0 \leq i \leq n\} \subseteq \mathcal{P}^n$$

We answer these questions in Theorem 1.4 below.

Write e_i , $0 \leq i \leq n$, for the i th standard basis vector in \mathbb{F}_p^{n+1} .

Proposition 1.2. *For any integers l, m, n , with $0 \leq l \leq n$,*

$$(8) \quad \left\{ \sum_i \binom{k-i}{l} e_i : m \leq k \leq m+l \right\}$$

is linear independent in \mathbb{F}_p^{n+1} .

Proposition 1.3. *The set*

$$(9) \quad \{P^i \chi P^{n-i} : l(n) + 1 \leq i \leq n\}$$

is linearly independent in \mathcal{P}^n .

Define a linear map

$$(10) \quad \mu : \mathbb{F}_p^{n+1} \rightarrow \mathcal{P}^n, \quad \mu e_i = P^i \chi P^{n-i}$$

Theorem 1.1 implies that if $l = l(n)$ the elements in (8) lie in $\ker \mu$, so Propositions 1.2 and 1.3 imply that (8) with $l = l(n)$ is a basis for $\ker \mu$ and that (9) is a basis for $V_n \subseteq \mathcal{P}^n$. Thus:

Theorem 1.4. *Any $l + 1$ consecutive relations from the set (6) with $l = l(n)$ form a basis of relations among the elements of $\{P^i \chi P^{n-i} : 0 \leq i \leq n\}$. The set $\{P^i \chi P^{n-i} : l(n) + 1 \leq i \leq n\}$ is a basis for V_n .*

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2. INDEPENDENCE OF THE RELATIONS

We wish to show that (8) is a linearly independent set. Regard elements of \mathbb{F}_p^{n+1} as column vectors, and arrange the $l + 1$ vectors in (8) as columns in a matrix, which we claim is of rank $l + 1$. The top square portion is the mod p reduction of the $(l + 1) \times (l + 1)$ integral Toeplitz matrix $A_l(m)$ with (i, j) th entry

$$\binom{m + j - i}{l}, \quad 0 \leq i, j \leq l$$

Lemma 2.1. $\det A_l(m) = 1$.

Proof. By induction on m . Since $\binom{-1}{l} = (-1)^l$ and $\binom{-1+j}{l} = 0$ for $0 < j \leq l$, $A_l(-1)$ is lower triangular with determinant $((-1)^l)^{l+1} = 1$. Now we note the identity

$$BA_l(m) = A_l(m + 1)$$

where

$$B = \begin{bmatrix} \binom{l+1}{1} & -\binom{l+1}{2} & \cdots & (-1)^{l-1} \binom{l+1}{l} & (-1)^l \binom{l+1}{l+1} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

The matrix identity is an expression of the binomial identity

$$(11) \quad \sum_k (-1)^k \binom{l+1}{k} \binom{n-k}{l} = 0$$

(taking $n = m + 1 - j$ and $k = j + 1$). Since $\det B = 1$, the result follows for all $m \in \mathbb{Z}$. \square

For completeness, we note that (11) is the case $m = l + 1$ of:

Lemma 2.2. $\sum_k (-1)^k \binom{m}{k} \binom{n-k}{l} = \binom{n-m}{l-m}.$

Proof. The defining identity for binomial coefficients implies the case $m = 1$, and also that both sides satisfy the recursion $C(l, m, n) - C(l, m, n-1) = C(l, m+1, n)$. \square

3. INDEPENDENCE OF THE OPERATIONS

We will prove Proposition 1.3 by studying how $P^i \chi P^{n-i}$ pairs against elements in \mathcal{P}_* , the dual of the Hopf algebra of Steenrod reduced powers. According to Milnor [4], with our grading conventions

$$\mathcal{P}_* = \mathbb{F}_p[\xi_1, \xi_2, \dots], \quad |\xi_j| = \frac{p^j - 1}{p - 1},$$

and

$$(12) \quad \Delta \xi_k = \sum_{i+j=k} \xi_i^{2^j} \otimes \xi_j.$$

For a finitely nonzero sequence of nonnegative integers $R = (r_1, r_2, \dots)$ write $\xi^R = \xi_1^{r_1} \xi_2^{r_2} \dots$ and let $\|R\| = r_1 + pr_2 + p^2r_3 + \dots$ and

$$|R| = |\xi^R| = r_1 + \left(\frac{p^2 - 1}{p - 1} \right) r_2 + \left(\frac{p^3 - 1}{p - 1} \right) r_3 + \dots.$$

The following clearly implies Proposition 1.3.

Proposition 3.1. *For any integer $n > 0$ there exist sequences $R_{n,j}$, $0 \leq j \leq n - l(n) - 1$, such that $|R_{n,j}| = n$ and*

$$\langle P^i \chi P^{n-i}, \xi^{R_{n,j}} \rangle = \begin{cases} \pm 1 & \text{for } i = n - j \\ 0 & \text{for } i > n - j. \end{cases}$$

The starting point in proving this is the following result of Milnor [4].

Lemma 3.2 ([4], Lemma 10). $\langle \chi P^n, \xi^R \rangle \neq 0$ for all sequences R with $|R| = n$.

In the basis of \mathcal{P} dual to the monomial basis of \mathcal{P}_* , the element corresponding to ξ_1^i is P^i . Since the diagonal in \mathcal{P}_* is dual to the product in \mathcal{P} , it follows from (12) that

$$\langle P^i \chi P^{n-i}, \xi^R \rangle = \begin{cases} \pm 1 & \text{for } i = \|R\| \\ 0 & \text{for } i > \|R\|. \end{cases}$$

So we wish to construct sequences $R_{n,j}$, for $l(n) + 1 \leq j \leq n$, such that $|R_{n,j}| = n$ and $\|R_{n,j}\| = j$. We deal first with the case $j = l(n) + 1$.

Proposition 3.3. *For any $n \geq 0$ there is a sequence $M = (m_1, m_2, \dots)$ such that*

- (1) $|M| = n$,
- (2) $0 \leq m_i \leq p$ for all i , and
- (3) If $m_j = p$ then $m_i = 0$ for all $i < j$.

For any such sequence, $\|M\| = l(n) + 1$.

Proof. Give the set of sequences of dimension n the right-lexicographic order. We claim that the maximal sequence satisfies the hypotheses.

Suppose that $R = (r_1, r_2, \dots)$ does not satisfy the hypotheses. If $r_1 > p$ then the sequence $(r_1 - (p + 1), r_2 + 1, r_3, \dots)$ is larger. If $r_j > p$, with $j > 1$, then the sequence $(r_1, \dots, r_{j-2}, r_{j-1} + p, r_j - (p + 1), r_{j+1} + 1, r_{j+2}, \dots)$ is larger. This proves (2). To prove (3), suppose that $r_j = p$ with $j > 1$, and suppose that some earlier entry is nonzero. Let $i = \min\{k : r_k > 0\}$. If $i = 1$, then the sequence $(r_1 - 1, r_2, \dots, r_{j-1}, 0, r_{j+1} + 1, r_{j+2}, \dots)$ is larger. If $i > 1$, then S with $s_k = 0$ for $k < i - 1$ and $i \leq k \leq j$, $s_{i-1} = p$, $s_{j+1} = r_{j+1} + 1$, and $s_k = r_k$ for $k > j + 1$, is larger.

Let M be a sequence satisfying (1)–(3), and write $l = \|M\| - 1$. To see that $l = l(n)$ we must show that

$$(13) \quad p(l + 1) - \alpha(l + 1) \geq (p - 1)n$$

and

$$(14) \quad pl - \alpha(l) < (p - 1)n.$$

The *excess* $e(R)$ is the sum of the entries in R , so that $p\|R\| - e(R) = (p - 1)|R|$. The p -adic representation of a number minimizes excess, so for any sequence R we have $e(R) \geq \alpha(\|R\|)$ and hence $p\|R\| - \alpha(\|R\|) \geq (p - 1)|R|$: so (13) holds for any sequence.

To see that (14) holds for M , let $j = \min\{i : m_i > 0\}$, so that $(p - 1)n = (p^j - 1)m_j + (p^{j+1} - 1)m_{j+1} + \dots$ and $l + 1 = p^{j-1}m_j + p^j m_{j+1} + \dots$. The hypotheses imply that l has p -adic expansion

$$(1 + \dots + p^{j-2})(p - 1) + p^{j-1}(m_j - 1) + p^j m_{j+1} + \dots,$$

so

$$\alpha(l) = (j - 1)(p - 1) + (m_j - 1) + m_{j+1} + \dots$$

from which we deduce

$$pl - \alpha(l) = (p - 1)(n - j) < (p - 1)n.$$

Corollary 3.4. *The function $l(n)$ is weakly increasing.*

Proof. Let M be a sequence satisfying the conditions of Proposition 3.3, and note that the sequence $R = (1, 0, 0, \dots) + M$ has $|R| = n + 1$ and $\|R\| = \|M\| + 1 = l(n) + 2$. If p does not occur in M , then R satisfies the hypotheses of the proposition (in degree $n + 1$) and hence $l(n) + 1 \leq l(n + 1) + 1$. If p does occur in M , then the moves described above will lead to a sequence M' satisfying the hypotheses. None of the moves decrease $\|-\|$, so $l(n) + 1 \leq l(n + 1) + 1$. \square

Remark 3.5. Properties (1)–(3) in fact determine M uniquely.

Proof of Proposition 3.1. Define $R_{n,l(n)+1}$ to be a sequence M as in Proposition 3.3. Then inductively define

$$R_{n,j} = (1, 0, 0, \dots) + R_{n-1,j-1} \quad \text{for } l(n) + 1 < j \leq n.$$

This makes sense by monotonicity of $l(n)$, and the elements clearly satisfy $|R_{n,j}| = n$ and $\|R_{n,j}\| = j$. This completes the proof. \square

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